The Journal of Gifu Keizai University, Vol. 37, No. 1 (2003)

COHOMOLOGY OF QUADRATIC NAMBU-POISSON TENSOR

by

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1. Introduction

In the present paper, we compute Nambu-Poisson cohomology in the case of quadratic Nambu-Poisson tensor. The notion of Nambu-Poisson cohomology was first introduced by R. Ibáñez *et al* [2]. Let us consider $\eta = (x^2 + y^2 + z^2 + u^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$, which is a Nambu-Poisson tensor of order 3 defined on $\mathbb{R}^4(x, y, z, u)$. To compute $H^*_{NP}(\mathbb{R}^4, \eta)$, we will essentially use the results of computations of $H^*_{NP}(\mathbb{R}^3, \eta')$, where $\eta' = (x^2 + y^2 + z^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$.

2. Computation of Nambu-Poisson Cohomology

2.1. **Notation and General Remarks.** First of all we review an equivalent cohomology to Nambu-Poisson cohomology, which is due to P. Monnier [3]. Let M be an m-dimensional C^∞-manifold with a volume form Ω. For $h \in C^\infty(M)$, we define the operator $d_h : \Omega^k(M) \to \Omega^{k+1}(M)$ by $d_h(\alpha) = hd\alpha - kdh \wedge \alpha$. It is easy to prove that $d_h \circ d_h = 0$. We denote by $H^*_h(M)$ the cohomology of this complex. Let η be an element of $\Gamma(\Lambda^m(TM))$. Recall that such η becomes always a Nambu-Poisson tensor [4]. Then P. Monnier proved the following [3].

This work has been supported by grant-in-aid of Group Researches of Gifu-Keizai University (2002–2004). We would like to express our deep thanks to Y. Agaoka for helpful discussions.

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Proposition 2.1. *If we put* $h = i_\eta \Omega$, then $H^*_{NP}(M, \eta)$ *is isomorphic to* $H^*_h(M)$.

It is easy to see that if g is a function on M which does not vanish on M, then the cohomologies $H^*_h(M)$ and $H^*_{hg}(M)$ are isomorphic.

Throughout this paper, we will use the following notations:

- $\mathcal{F}$ is the algebra of real-valued C^∞ functions on $\mathbb{R}^4(x, y, z, u)$;
- $\mathcal{F}'$ is the algebra of real-valued C^∞ functions on $\mathbb{R}^3(x, y, z)$;
- $\chi(\mathbb{R}^4)$ is the $\mathcal{F}$-module of vector fields on $\mathbb{R}^4$;
- $\chi'(\mathbb{R}^4) = \left\{A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} \mid A, B, C \in \mathcal{F}\right\}$;
- $f = x^2 + y^2 + z^2 + u^2$;
- $f' = x^2 + y^2 + z^2$;
- $\Omega^k =$ the space of k-forms on $\mathbb{R}^4$;
- $\Omega'_1 = \{Adx + Bdy + Cdz \mid A, B, C \in \mathcal{F}\}$;
- $\Omega'_2 = \{Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy \mid A, B, C \in \mathcal{F}\}$;
- $\Omega'_3 = \{Adx \wedge dy \wedge dz \mid A \in \mathcal{F}\}$.

If we choose $\omega = dx \wedge dy \wedge dz$ as the volume form on $\mathbb{R}^3$, then we have $f' = i_{\eta'}\omega$. First we compute $H^*_{NP}(\mathbb{R}^3, \eta')$, which is isomorphic to $H^*_{f'}(\mathbb{R}^3)$ by Proposition 2.1. In the formal category (i.e. all coefficients of differential forms are formal power series), the following results were obtained by P. Monnier [3].

Proposition 2.2. *In the formal category,* $H^0_{f'} \cong \mathbb{R}$, $H^1_{f'} \cong \mathbb{R}$, $H^2_{f'} \cong 0$ *and* $H^3_{f'} \cong \mathbb{R}$.

We want to compute $H^*_{f'}$ in the C^∞-category, and we will show that Proposition 2.2 still holds even in the C^∞-category. First it is clear that $H^0_{f'} \cong \mathbb{R}$. R. Ibáñez *et al* [2] proved independently of P. Monnier [3] that $H^1_{f'} \cong \mathbb{R}$. Hence it only remains to compute $H^2_{f'}$ and $H^3_{f'}$. To compute them, we use Proposition 2.2.

Let β be a 2-cocycle. Then by definition, β satisfies $f'd\beta = 2df' \wedge \beta$. Denote by $[\beta]$ the formal Taylor expansion of β at the origin. Then by Proposition 2.2, there exists a formal 1-form $[\alpha]$ such that $[\beta] = f'd[\alpha] - df' \wedge [\alpha]$. Hence we can find a 1-form α, whose formal Taylor expansion at the origin is $[\alpha]$. Put $\beta' = \beta - (f'd\alpha - df' \wedge \alpha)$. Then β' is flat (i.e. $[\beta'] = 0$) and satisfies $f'd\beta' = 2df' \wedge \beta'$. $\frac{\beta'}{f'^2}$ is also flat and $d\left(\frac{\beta'}{f'^2}\right) = \frac{1}{f'^3}(f'd\beta' - 2df' \wedge \beta') = 0$. Hence there exists a flat 1-form $\tilde{\alpha}$ such that $\frac{\beta'}{f'^2} = d\tilde{\alpha}$. Put $\tilde{\alpha} = \frac{\alpha'}{f'}$. Then α' is a flat 1-form, and we get $\beta'=f'^2d\tilde{\alpha}=f'd\alpha'-df'\wedge\alpha'$. Finally we have

$$\beta=f'd(\alpha+\alpha')-df'\wedge(\alpha+\alpha').$$

This means $H^2_{f'}=0$.

Next let us compute $H^3_{f'}$. The space of 3-cocycles $Z^3_{f'}$ is clearly isomorphic to $\mathcal{F}'$. And the space of 3-coboundaries $B^3_{f'}$ is isomorphic to the following space $\mathcal{F}_1$.

$$\mathcal{F}_1=\left\{f'\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right)-4(xA+yB+zC);\ A,\ B,\ C\in\mathcal{F}'\right\}$$

Lemma 2.3. *Let $\mathcal{I}$ be the subspace of $\mathcal{F}'$ consisting of functions which are flat at the origin. Then $\mathcal{I}\subset\mathcal{F}_1$.*

Proof. For $q\in\mathcal{I}$, put

$$A=(f')^2\int\frac{q}{(f')^3}dx,\quad B=0,\quad C=0.$$

Then $f'(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z})-4(xA+yB+zC)=q$. Hence we have that $q\in\mathcal{F}_1$. □

Denote by F' (resp. F_1) the formal algebra corresponding to $\mathcal{F}'$ (resp. $\mathcal{F}_1$). Let T be a mapping from $\mathcal{F}'$ to F', where $T(h)$ is the formal Taylor expansion of h at the origin. Let $\pi: F'\to F'/F_1$ be the canonical projection, and put $\tilde{T}=\pi\circ T$. Then $\tilde{T}$ is a surjective linear mapping and it is clear that ker $\tilde{T}=\mathcal{F}_1$ by Lemma 2.3. Since $F'/F_1\cong\mathbb{R}$ by Proposition 2.2, we get that

$$H^3_{f'}\cong\mathcal{F}'/\mathcal{F}_1\cong F'/F_1\cong\mathbb{R}.$$

Thus we obtained the following proposition.

Proposition 2.4. *In C^∞-category, it still holds that $H^0_{f'}\cong\mathbb{R}$, $H^1_{f'}\cong\mathbb{R}$, $H^2_{f'}=0$ and $H^3_{f'}\cong\mathbb{R}$.*

For the Nambu-Poisson tensor $\eta=f\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$ defined on $\mathbb{R}^4$, we know that

$$\sharp_2(\Omega^2)=\{fX|X\in\chi'(\mathbb{R}^4)\}.$$

$\sharp_2(\Omega^2)$ is denoted by $\mathfrak{g}$, which is isomorphic to $\Omega^2/\ker\sharp_2$. Note also that $\Omega^2/\ker\sharp_2$ is isomorphic to Ω'_2. $\mathfrak{g}$ is, of course, a Lie subalgebra of $\chi(\mathbb{R}^4)$.

Since $H^0_{NP}(\mathbb{R}^4, \eta) = \{g \in \mathcal{F} \mid Xg = 0 \text{ for all } X \in \mathfrak{g}\}$, it is clear that $H^0_{NP}(\mathbb{R}^4, \eta) \cong C^\infty(\mathbb{R})$.

In computing Nambu-Poisson cohomology, we use Proposition 2.4. To do this, we need the formal Taylor expansion of a function $A \in \mathcal{F}$ with respect to the variable u, which is denoted by $\overline{A}$. In other words, three variables x, y and z are regarded as parameters. And we say that $\overline{A}$ is the *u-formal Taylor expansion* of A. This terminology will be also used for differential forms and vector fields. Thus we can express $\overline{A}$ (similarly $\overline{B}$ and $\overline{C}$) as follows.

$$
\begin{cases} \overline{A} = a_0 + ua_1 + u^2 a_2 + \cdots, \\ \overline{B} = b_0 + ub_1 + u^2 b_2 + \cdots, \\ \overline{C} = c_0 + uc_1 + u^2 c_2 + \cdots, \end{cases} \tag{1}
$$

where a_k, b_k, $c_k \in \mathcal{F}'$.

To compute $H^k_{NP}(\mathbb{R}^4, \eta)$, $k \geq 1$, let us define a linear mapping $d' : \mathcal{F} \rightarrow \Omega'_1$ by

$$
d'g = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz .
$$

This operator d' is naturally extended to a linear mapping from Ω'_k to Ω'_{k+1}. Moreover we define $d'_f : \Omega'_k \rightarrow \Omega'_{k+1}$ by

$$
d'_f(\alpha) = fd'\alpha - kd'f \wedge \alpha, \quad \alpha \in \Omega'_k .
$$

Then $d'_f \circ d'_f = 0$, and we denote by $H^*_{d'_f}$ the cohomology space with respect to d'_f.

If we define $b : \chi'(\mathbb{R}^4) \rightarrow \Omega'_2$ by $b(X) = i(X) dx \wedge dy \wedge dz$, then we obtain that $\sharp_2(b(X)) = fX$ and that $\sharp_2(\{b(X), b(Y)\}) = [\sharp_2(b(X)), \sharp_2(b(Y))] = [fX, fY]$.

Following the similar method of P. Monnier [3], if $\phi : C^k(\Omega'_2, \mathcal{F}) \rightarrow \Omega'_k$ is defined by

$$
\phi(c^k)(X_1, \cdots X_k) = c^k(b(X_1), \cdots b(X_k)), X_1, \cdots X_k \in \chi'(\mathbb{R}^4),
$$

then ϕ is a linear isomorphism and we can prove the following.

Proposition 2.5. *The following diagram is commutative.*

$$
\begin{array}{ccc}
C^k(\Omega'_2, \mathcal{F}) & \xrightarrow{\ \phi\ } & \Omega'_k \\
\partial \downarrow & & \downarrow d'_f \\
C^{k+1}(\Omega'_2, \mathcal{F}) & \xrightarrow[\ \phi\ ]{} & \Omega'_{k+1}
\end{array}
$$

Hence $H^*_{NP}(\mathbb{R}^4, \eta) \cong H^*_{d'_f}$.

Proof. We prove only for the case $k=1$. For $c \in C^1(\Omega'_2, \mathcal{F})$, put $\phi(c) = \alpha$. For any $X, Y \in \chi'(\mathbb{R}^4)$, we can directly get

$$\{b(X), b(Y)\} = f \cdot b([X, Y]) - (Xf) \cdot b(Y) + (Yf) \cdot b(X),$$

from the definition of the bracket $\{,\}$ on Ω'_2. Using this equation, we have

$$\begin{aligned}
\phi(\partial c)(X, Y) &= (\partial c)(b(X), b(Y)) \\
&= fX \cdot c(b(Y)) - fY \cdot c(b(X)) - c(\{b(X), b(Y)\}) \\
&= fX \cdot \alpha(Y) - fY \cdot \alpha(X) - c(f \cdot b([X, Y]) \\
&\quad + (Xf) \cdot b(Y) - (Yf) \cdot b(X)) \\
&= fX \cdot \alpha(Y) - fY \cdot \alpha(X) - f\alpha([X, Y]) \\
&\quad - (Xf) \cdot \alpha(Y) + (Yf) \cdot \alpha(X) \\
&= f \cdot d'\alpha(X, Y) - (d'f \wedge \alpha)(X, Y) \\
&= (d'_f\alpha)(X, Y) = (d'_f \circ \phi(c))(X, Y).
\end{aligned}$$

Thus $\phi \circ \partial = d'_f \circ \phi$. □

2.2. **Computation of** $H^1_{NP}(\mathbb{R}^4, \eta)$. In this section, we compute $H^1_{NP}(\mathbb{R}^4, \eta)$. In order to do this, we have only to compute $H^*_{d'_f}$ by Proposition 2.5. The space of 1-coboundaries, which is denoted by B'_1, is the set of 1-forms $fd'g$, $g \in \mathcal{F}$. Let Z'_1 be the space of 1-cocycles. Then for $\alpha = Adx + Bdy + Cdz \in \Omega'_1$, α is an element of Z'_1 if and only if $fd'\alpha = d'f \wedge \alpha$. This equation is equivalent to the following three equations.

$$(2) \qquad \begin{cases} f \cdot \left(\dfrac{\partial B}{\partial x} - \dfrac{\partial A}{\partial y}\right) = 2xB - 2yA, \\[2ex] f \cdot \left(\dfrac{\partial C}{\partial y} - \dfrac{\partial B}{\partial z}\right) = 2yC - 2zB, \\[2ex] f \cdot \left(\dfrac{\partial A}{\partial z} - \dfrac{\partial C}{\partial x}\right) = 2zA - 2xC. \end{cases}$$

Note that u-formal Taylor expansion of α is written as $\bar{\alpha} = \alpha_0 + u\alpha_1 + u^2\alpha_2 + \cdots$, where $\alpha_p = a_p dx + b_p dy + c_p dz$, $a_p, b_p, c_p \in \mathcal{F}'$. And three equations (2) have the following formal expression.

$$
(3) \qquad \begin{cases} f\cdot\left(\dfrac{\partial \bar{B}}{\partial x}-\dfrac{\partial \bar{A}}{\partial y}\right)=2x\bar{B}-2y\bar{A}, \\ f\cdot\left(\dfrac{\partial \bar{C}}{\partial y}-\dfrac{\partial \bar{B}}{\partial z}\right)=2y\bar{C}-2z\bar{B}, \\ f\cdot\left(\dfrac{\partial \bar{A}}{\partial z}-\dfrac{\partial \bar{C}}{\partial x}\right)=2z\bar{A}-2x\bar{C}. \end{cases}
$$

Comparing constant terms of u in the both sides of (3), we have

$$
(4) \qquad \begin{cases} f'\cdot\left(\dfrac{\partial b_0}{\partial x}-\dfrac{\partial a_0}{\partial y}\right)=2xb_0-2ya_0, \\ f'\cdot\left(\dfrac{\partial c_0}{\partial y}-\dfrac{\partial b_0}{\partial z}\right)=2yc_0-2zb_0, \\ f'\cdot\left(\dfrac{\partial a_0}{\partial z}-\dfrac{\partial c_0}{\partial x}\right)=2za_0-2xc_0. \end{cases}
$$

These three equations (4) essentially appeared in computing $H^1_{NP}(\mathbb{R}^3, \eta' = f'\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z})$. By Proposition 2.4, $H^1_{NP}(\mathbb{R}^3, \eta')$ is isomorphic to $\mathbb{R}$. The generator of $H^1_{NP}(\mathbb{R}^3, \eta')$ is df' and this means that there exist a real number k_0 and a function $g_0 \in \mathcal{F}'$ such that

$$
(5) \qquad \begin{cases} a_0 = k_0\cdot 2x + f'\cdot\dfrac{\partial g_0}{\partial x}, \\ b_0 = k_0\cdot 2y + f'\cdot\dfrac{\partial g_0}{\partial y}, \\ c_0 = k_0\cdot 2z + f'\cdot\dfrac{\partial g_0}{\partial z}. \end{cases}
$$

Since $\alpha_0 = a_0dx + b_0dy + c_0dz$, we obtain that $\alpha_0 = k_0df' + f'dg_0$. Similarly if we compare the coefficients of u in the both sides of (3), we can get $\alpha_1 = k_1df' + f'dg_1$, where $k_1 \in \mathbb{R}$ and $g_1 \in \mathcal{F}'$. But if we compare the coefficients of u^2 in the both sides of (3), the situation is slightly different. In fact, we have

$$
(6) \qquad \begin{cases} f'\cdot\left(\dfrac{\partial b_2}{\partial x}-\dfrac{\partial a_2}{\partial y}\right)+\left(\dfrac{\partial b_0}{\partial x}-\dfrac{\partial a_0}{\partial y}\right)=2xb_2-2ya_2, \\ f'\cdot\left(\dfrac{\partial c_2}{\partial y}-\dfrac{\partial b_2}{\partial z}\right)+\left(\dfrac{\partial c_0}{\partial y}-\dfrac{\partial b_0}{\partial z}\right)=2yc_2-2zb_2, \\ f'\cdot\left(\dfrac{\partial a_2}{\partial z}-\dfrac{\partial c_2}{\partial x}\right)+\left(\dfrac{\partial a_0}{\partial z}-\dfrac{\partial c_0}{\partial x}\right)=2za_2-2xc_2. \end{cases}
$$

These equations (6) can be rewritten as follows.

$$(7)\quad \begin{cases} f'\left(\dfrac{\partial\left(b_2-\frac{\partial g_0}{\partial y}\right)}{\partial x}-\dfrac{\partial\left(a_2-\frac{\partial g_0}{\partial x}\right)}{\partial y}\right)=2x\left(b_2-\dfrac{\partial g_0}{\partial y}\right)-2y\left(a_2-\dfrac{\partial g_0}{\partial x}\right), \\ f'\left(\dfrac{\partial\left(c_2-\frac{\partial g_0}{\partial z}\right)}{\partial y}-\dfrac{\partial\left(b_2-\frac{\partial g_0}{\partial y}\right)}{\partial z}\right)=2y\left(c_2-\dfrac{\partial g_0}{\partial z}\right)-2z\left(b_2-\dfrac{\partial g_0}{\partial y}\right), \\ f'\left(\dfrac{\partial\left(a_2-\frac{\partial g_0}{\partial x}\right)}{\partial z}-\dfrac{\partial\left(c_2-\frac{\partial g_0}{\partial z}\right)}{\partial x}\right)=2z\left(a_2-\dfrac{\partial g_0}{\partial x}\right)-2x\left(c_2-\dfrac{\partial g_0}{\partial z}\right). \end{cases}$$

Thus we can apply Proposition 2.4 to (7), and we have that there exist a real number k_2 and $g_2 \in \mathcal{F}'$ such that

$$(8)\quad \begin{cases} a_2-\dfrac{\partial g_0}{\partial x}=k_2\cdot 2x+f'\dfrac{\partial g_2}{\partial x}, \\ b_2-\dfrac{\partial g_0}{\partial y}=k_2\cdot 2y+f'\dfrac{\partial g_2}{\partial y}, \\ c_2-\dfrac{\partial g_0}{\partial z}=k_2\cdot 2z+f'\dfrac{\partial g_2}{\partial z}. \end{cases}$$

Hence $\alpha_2=k_2df'+f'dg_2+dg_0$. By the same methods, we know that each α_p, ($p \geq 3$) has the form $\alpha_p=k_pdf'+f'dg_p+dg_{p-2}$, where $k_p\in\mathbb{R}$ and $g_{p-2}, g_p\in\mathcal{F}'$. These mean that $\bar{\alpha}$ has the following expression. Note that $df'=d'f$ and that $f'+u^2=f$.

$$\bar{\alpha}=(k_0+k_1u+k_2u^2+\cdots)d'f+f\cdot d'(g_0+ug_1+u^2g_2+\cdots).$$

To obtain the final result, we need the following lemma, which is a generalization of E. Borel theorem. This will be proved in the analogous way as K. Abe and K. Fukui, Lemma 4.4 [1]. (See also R. Narasimhan [5], §1.5.2 and §1.5.3.) We put $\vec{r}=(x, y, z, u)$ and $|\vec{r}|=\sqrt{x^2+y^2+z^2+u^2}$. Then a function $F(\vec{r}) \in C^\infty(\mathbb{R}^4)$ is said to be *m-flat* as a function of u at $(x, y, z, 0)$ if $\frac{\partial^\alpha}{\partial u^\alpha}F(x, y, z, 0)=0$ for $\alpha\leq m$.

Lemma 2.6. *For each integer $p\geq 0$, let $c_p(x, y, z)\in C^\infty(\mathbb{R}^3)$. Then there exists $G(\vec{r})\in C^\infty(\mathbb{R}^4)$ such that the partial derivatives with respect to the last variable of G at any point $(x, y, z, 0)\in\mathbb{R}^4$ are*

$$\frac{\partial^p G}{\partial u^p}(x, y, z, 0) = p!c_p(x, y, z) \quad p \geq 0.$$

Proof. Let $T_m(\vec{r}) = \sum_{p=0}^{m} c_p(x, y, z) u^p$ for $\vec{r} \in \mathbb{R}^4$. Let $H(\vec{r}) \in C^\infty(\mathbb{R}^4)$ such that $H(\vec{r}) = 0$ for $|\vec{r}| \leq 1/2$, $H(\vec{r}) = 1$ for $|\vec{r}| \geq 1$ and $H(\vec{r}) \geq 0$ for any $\vec{r} \in \mathbb{R}^4$. For a positive number δ, put

$$g_\delta(\vec{r}) = H\left(\frac{\vec{r}}{\delta}\right)(T_{m+1}(\vec{r}) - T_m(\vec{r})).$$

Clearly $g_\delta \in C^\infty(\mathbb{R}^4)$ and vanishes near 0. Moreover $T_{m+1} - T_m$ is m-flat as a function of u at any point $(x, y, z, 0)$. Hence as in the proof of Lemma 1.5.2 [5], there exists a positive number δ_m such that

$$\sum_{p=0}^{m} \frac{1}{p!} \left| \frac{\partial^p}{\partial u^p} (g_{\delta_m} - (T_{m+1} - T_m))(\vec{r}) \right| < 2^{-m}.$$

Put $g_m = g_{\delta_m}$. If we define

$$G = T_0 + \sum_{m=0}^{\infty} (T_{m+1} - T_m - g_m),$$

then as in the proof of Lemma 1.5.3 [5], we get that the function G is the desired function. □

By Lemma 2.6, we obtain that there exist a C^∞-function $k(u)$ and a C^∞-function $g(x, y, z, u)$ such that $\overline{k(u)} = k_0 + k_1 u + k_2 u^2 + \cdots$, and $\overline{g(x, y, z, u)} = g_0 + u g_1 + u^2 g_2 + \cdots$. Put $\alpha' = k(u) d'f + f d'g$, and put $\alpha - \alpha' = \alpha_f$. Then α_f is a 1-cocycle and it satisfies $\overline{\alpha_f} = 0$ (u-flat 1-form). Let $k_1(u)$ be a flat function of one variable u. Then $(\alpha_1 - k_1(u) d'f)/f$ is a well-defined 1-form on $\mathbb{R}^4$, and it satisfies

$$d'\left(\frac{\alpha_f - k_1(u) d'f}{f}\right) = \frac{1}{f^2}(f d'\alpha_f - d'f \wedge (\alpha_f - k_1(u) d'f)) = 0.$$

Hence, as is easily seen, there exists a flat function $\tilde{g}(x, y, z, u)$ such that $(\alpha_f - k_1(u) d'f)/f = d'\tilde{g}$. And we obtain that $\alpha \in Z_1'$ has the following form:

$$\alpha = \alpha_f + \alpha' = (k(u) + k_1(u)) d'f + f d'(g + \tilde{g}).$$

α is, by definition, cohomologous to $(k(u) + k_1(u)) d'f$. Moreover $l(u) d'f$ is contained in B_1' if and only if $l(u)$ is a flat function at $u = 0$. In fact, note that in this case $l(u) \log f$ is a C^∞-function and it holds that $l(u) d'f = f d'(l(u) \log f) \in B_1'$. Thus we obtain that $H^1_{NP}(\mathbb{R}^4, \eta)$ is isomorphic to $\mathbb{R}[[u]]$, which is the space of formal power series of one variable u.

2.3. Computation of $H^2_{NP}(\mathbb{R}^4, \eta)$. We will compute $H^2_{NP}(\mathbb{R}^4, \eta)$. By Proposition 2.5, we will compute $H^2_{d'_f}$. Every computation proceeds in the analogous way as the case of $H^1_{d'_f}$. The space of 2-coboundaries B'_2 is, by definition, the set of 2-forms $d'_f\gamma = fd'\gamma - d'f \wedge \gamma$, $\gamma \in \Omega'_1$. Let Z'_2 be the space of 2-cocycles. Then for $\beta = A dy \wedge dz + B dz \wedge dx + C dx \wedge dy \in \Omega'_2$, β is an element of Z'_2 if and only if $fd'\beta = 2d'f \wedge \beta$. This is equivalent to

$$f \cdot \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) = 4(xA + yB + zC). \tag{9}$$

u-formal Taylor expansion (with respect to u) of β is written as $\bar{\beta} = \beta_0 + u\beta_1 + u^2\beta_2 + \cdots$, where $\beta_p = a_p dy \wedge dz + b_p dz \wedge dx + c_p dx \wedge dy$, $a_p, b_p, c_p \in \mathcal{F}'$. Then the equation (9) has the following u-formal Taylor expansion.

$$f \cdot \left(\frac{\partial \bar{A}}{\partial x} + \frac{\partial \bar{B}}{\partial y} + \frac{\partial \bar{C}}{\partial z} \right) = 4(x\bar{A} + y\bar{B} + z\bar{C}). \tag{10}$$

Comparing constant terms with respect to u in the both sides of (10), we have

$$f' \cdot \left(\frac{\partial a_0}{\partial x} + \frac{\partial b_0}{\partial y} + \frac{\partial c_0}{\partial z} \right) = 4(xa_0 + yb_0 + zc_0). \tag{11}$$

This is equivalent to $d_{f'}\beta_0 = 0$ for $\beta_0 = a_0 dy \wedge dz + b_0 dz \wedge dx + c_0 dx \wedge dy$. Recall that $H^2_{NP}(\mathbb{R}^3, \eta') = 0$ by Proposition 2.4. In other words, if $d_{f'}\beta_0 = 0$, then β_0 must be a coboundary. This means that we can find a 1-form α_0 such that $\beta_0 = f'd\alpha_0 - df' \wedge \alpha_0$.

Comparing the coefficients of u in the both sides of (10), we can also find a 1-form α_1 such that $\beta_1 = f'd\alpha_1 - df' \wedge \alpha_1$. Moreover if $p \geq 2$ we can find p-form α_p such that $\beta_p = f'd\alpha_p - d'f \wedge \alpha_p + d\alpha_{p-2}$. u-formal Taylor expansion of β is as follows.

$$\begin{aligned}
\bar{\beta} &= \sum_{p=0}^{\infty} u^p \beta_p \\
&= \sum_{p=0}^{\infty} u^p (f'd\alpha_p - d'f \wedge \alpha_p) + \sum_{p=0}^{\infty} u^{p+2} d\alpha_p \\
&= \sum_{p=0}^{\infty} u^p (f'd\alpha_p - df' \wedge \alpha_p + u^2 d\alpha_p) \\
&= \sum_{p=0}^{\infty} u^p (fd\alpha_p - d'f \wedge \alpha_p)
\end{aligned}$$

$$= fd'\Big(\sum_{p=0}^{\infty} u^p \alpha_p\Big) - d'f \wedge \Big(\sum_{p=0}^{\infty} u^p \alpha_p\Big).$$

Put $\hat{\alpha} = \sum_{p=0}^{\infty} u^p \alpha_p$. Then $\bar{\beta} = fd'\hat{\alpha} - d'f \wedge \hat{\alpha}$. By Lemma 2.6, there exists a 1-form $\alpha' \in \Omega_1'$ such that $\bar{\alpha}' = \hat{\alpha}$. Put $\beta' = fd'\alpha' - d'f \wedge \alpha'$. Then $\bar{\beta} = \bar{\beta}'$ and hence if we put $\tilde{\beta} = \beta - \beta'$, then $\tilde{\beta}$ is a flat 2-form of Ω_2'. Moreover it is easy to see that $fd'\tilde{\beta} = 2d'f \wedge \tilde{\beta}$, which means $\tilde{\beta} \in Z_2'$. Then by the same method as the proof of $H_{f'}^2 = 0$ (C^{∞}-case), we can prove that there exists a flat 1-form α_2 such that $\tilde{\beta} = fd'\alpha_2 - d'f \wedge \alpha_2$. Hence β has the following form:

$$\beta = \beta' + \tilde{\beta} = fd'(\alpha' + \alpha_2) - d'f \wedge (\alpha' + \alpha_2),$$

and thus $\beta \in B_2'$. Hence we get $H_{NP}^2(\mathbb{R}^4, \eta) = 0$.

2.4. **Computation of** $H_{NP}^3(\mathbb{R}^4, \eta)$. Let Z_3' be the space of 3-cocycles. Since $\Omega_4' = 0$, it holds that $Z_3' = \Omega_3'$. Hence Z_3' is isomorphic to $\mathcal{F}$. Let B_3' be the space of 3-coboundaries. Then every element of B_3' is written as

$$\begin{aligned} d_f'\beta &= fd'\beta - 2d'f \wedge \beta \\ &= \Big\{ f\Big(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\Big) - 4(xA + yB + zC)\Big\}\, dx \wedge dy \wedge dz, \end{aligned}$$

where $\beta = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$ is an arbitrary element of Ω_2'.

Put $\mathcal{B} = \{f\,(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}) - 4(xA + yB + zC) \mid A, B, C \in \mathcal{F}\}$. Then, by Proposition 2.5, $H_{NP}^3(\mathbb{R}^4, \eta)$ is isomorphic to $\mathcal{F}/\mathcal{B}$.

Lemma 2.7. *Put* $\mathcal{I} = \{h \in \mathcal{F} \mid \frac{\partial^p h}{\partial u^p}(x, y, z, 0) = 0,\ p \geq 0\}$. *i.e., each element* h *of* $\mathcal{I}$ *is u-flat. Then* $\mathcal{I} \subset \mathcal{B}$.

Proof. For $h \in \mathcal{I}$, it is clear that h/f^3 is an element of $\mathcal{F}$. Put $A = f^2 \int \frac{h}{f^3}\, dx$, $B = 0$ and $C = 0$. Then we have

$$f\Big(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\Big) - 4(xA + yB + zC) = h.$$

Hence $h \in \mathcal{B}$. □

Put $\hat{F} = \{\bar{A} \mid A \in \mathcal{F}\}$ and $\hat{B} = \{\bar{A} \mid A \in \mathcal{B}\}$. We also denote by $\mathcal{F}_0'$ the subspace of functions $g(x, y, z) \in \mathcal{F}'$ with $g(0, 0, 0) = 0$.

Proposition 2.8. $\hat{F}/\hat{B} \cong \mathbb{R}[[u]]$.

Proof. For any element $g = f(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}) - 4(xA + yB + zC) \in \mathcal{B}$, its u-formal Taylor expansion is

$$\hat{B} \ni \bar{g} = f\Big(\frac{\partial \bar{A}}{\partial x} + \frac{\partial \bar{B}}{\partial y} + \frac{\partial \bar{C}}{\partial z}\Big) - 4(x\bar{A} + y\bar{B} + z\bar{C})$$
$$= \sum_{p=0}^{\infty}\Big[u^p\Big\{f'\Big(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}\Big) - 4(xa_p + yb_p + zc_p)\Big\}$$
$$+ u^{p+2}\Big(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}\Big)\Big].$$

Put $g_p = f'(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}) - 4(xa_p + yb_p + zc_p)$ and $h_p = \frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}$ for non-negative integer p. Then every $\bar{g} \in \hat{B}$ has the following expression.

$$\bar{g} = (g_0 + u^2 h_0) + u(g_1 + u^2 h_1) + \cdots + u^p(g_p + u^2 h_p) + \cdots.$$

First recall that $H^3_{NP}(\mathbb{R}^3, \eta') \cong \mathbb{R}$ by Proposition 2.4. Hence for any non-negative integer p, it holds that

$$\{g_p \mid a_p, b_p, c_p \in \mathcal{F}'\} = \mathcal{F}'_0.$$

If we put $W_p = \{g_p + u^2 h_p \mid a_p, b_p, c_p \in \mathcal{F}'\}$, then $\bar{g}$ is contained in $W_0 + uW_1 + \cdots + u^p W_p + \cdots$. Note that h_p is not completely determined by g_p. To show this precisely, let us consider the following linear partial differential equation with three unknown functions $a, b, c \in \mathcal{F}'$.

$$(*) \qquad f'\Big(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\Big) - 4(xa + yb + zc) = 0.$$

We define a subspace $\mathcal{F}''_0$ of $\mathcal{F}'$ by

$$\mathcal{F}''_0 = \Big\{\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \mid \text{a triplet } (a, b, c) \text{ is a solution of } (*)\Big\}.$$

Since (a, b, c) is a solution of the differential equation $(*)$, there exist three functions $A, B, C \in \mathcal{F}'$ such that

$$(12) \qquad \begin{cases} a = f'(Cy - Bz) + 2(zB - yC), \\ b = f'(Az - Cx) + 2(xC - zA), \\ c = f'(Bx - Ay) + 2(yA - xB). \end{cases}$$

Recall that this fact is equivalent to $H^2_{f'} = 0$. Put $h = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}$. If h is an element of $\mathcal{F}''_0$, then it is clear that h vanishes at the origin and hence $h \in \mathcal{F}'_0$. Thus $\mathcal{F}''_0$ becomes a subspace of $\mathcal{F}'_0$.

Let

$$g_p = f'\Big(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}\Big) - 4(xa_p + yb_p + zc_p)$$

$$= f'\Big(\frac{\partial a'_p}{\partial x} + \frac{\partial b'_p}{\partial y} + \frac{\partial c'_p}{\partial z}\Big) - 4(xa'_p + yb'_p + zc'_p)$$

for two triplets (a_p, b_p, c_p) and (a'_p, b'_p, c'_p). Then we have

$$f'\Big(\frac{\partial(a_p - a'_p)}{\partial x} + \frac{\partial(b_p - b'_p)}{\partial y} + \frac{\partial(c_p - c'_p)}{\partial z}\Big) - 4\{x(a_p - a'_p) + y(b_p - b'_p) + z(c_p - c'_p)\} = 0.$$

Hence

$$h_p - h'_p = \frac{\partial(a_p - a'_p)}{\partial x} + \frac{\partial(b_p - b'_p)}{\partial y} + \frac{\partial(c_p - c'_p)}{\partial z}$$

is an element of $\mathcal{F}''_0$, where $h'_p = \frac{\partial a'_p}{\partial x} + \frac{\partial b'_p}{\partial y} + \frac{\partial c'_p}{\partial z}$.

We denote by $[h_p]$ a coset of h_p. Namely $[h_p]$ is an element of $\mathcal{F}'/\mathcal{F}''_0$. Using this expression, we know that $[h_p]$ is uniquely determined by g_p. Since the set $\{g_0 + ug_1 + u^2 g_2 + \cdots \mid g_k \in \mathcal{F}'_0\}$ spans $\mathbb{R}[[u]]\mathcal{F}'_0$, and since $u^p\mathcal{F}''_0$ is contained in $\mathbb{R}[[u]]\mathcal{F}'_0$, we can regard W_p as

$$W_p = \{g_p + u^2[h_p] \mid g_p \in \mathcal{F}'_0\}.$$

Let $\phi : W_p \longrightarrow \mathcal{F}'_0$ be a surjective linear mapping defined by $\phi(g_p + u^2[h_p]) = g_p$. Then it is clear that $g_p = 0$ means $[h_p] = 0$. Hence ϕ is injective. Thus we obtain that $\hat{B} \cong \mathbb{R}[[u]]\mathcal{F}'_0$.

Since

$$\begin{aligned}\hat{F} &= \mathcal{F}' + u\mathcal{F}' + u^2\mathcal{F}' + \cdots \\ &= (\mathbb{R} + \mathcal{F}'_0) + u(\mathbb{R} + \mathcal{F}'_0) + u^2(\mathbb{R} + \mathcal{F}'_0) + \cdots \\ &= \mathbb{R}[[u]] \oplus \mathbb{R}[[u]]\mathcal{F}'_0,\end{aligned}$$

we obtain that $\hat{F}/\hat{B} \cong \mathbb{R}[[u]]$. □

Let $T : \mathcal{F} \to \hat{F}$ be a linear mapping defined by $T(A) = \bar{A}$. For any $q \in T^{-2}(\hat{B})$, there exists $Q \in \hat{B}$ such that $T(q) = Q$. On the other hand, since

$T(\mathcal{B}) = \hat{B}$, there exists $q_1 \in \mathcal{B}$ such that $T(q_1) = Q$. Hence $q - q_1 \in \mathcal{I}$. By Lemma 2.7, we have $q \in \mathcal{B}$, and hence $T^{-1}(\hat{B}) = \mathcal{B}$. Thus by Proposition 2.8,

$$\mathcal{F}/\mathcal{B} \cong \hat{F}/\hat{B} \cong \mathbb{R}[[u]].$$

Now we summarize the results obtained in this section.

Theorem 2.9. *Let* $\eta = (x^2 + y^2 + z^2 + u^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ *be a Nambu-Poisson tensor on* $\mathbb{R}^4(x, y, z, u)$. Then

$$\begin{aligned} &H^0_{NP}(\mathbb{R}^4, \eta) \cong C^\infty(\mathbb{R}), \\ &H^1_{NP}(\mathbb{R}^4, \eta) \cong \mathbb{R}[[u]], \\ &H^2_{NP}(\mathbb{R}^4, \eta) = 0, \\ &H^3_{NP}(\mathbb{R}^4, \eta) \cong \mathbb{R}[[u]], \\ &H^k_{NP}(\mathbb{R}^4, \eta) = 0, \quad k \geq 4. \end{aligned}$$

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